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On the geometry of four-qubit invariants

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Abstract

The geometry of four-qubit entanglement is investigated. We replace some of the polynomial invariants for four qubits introduced recently by new ones of direct geometrical meaning. It is shown that these invariants describe four points, six lines and four planes in complex projective space \mathbf{CP}^3 . For the generic entanglement class of stochastic local operations and classical communication they take a very simple form related to the elementary symmetric polynomials in four complex variables. Moreover, suitable powers of their magnitudes are entanglement monotones that fit nicely into the geometric set of n -qubit ones related to Grassmannians of l -planes found recently. We also show that in terms of these invariants the hyperdeterminant of order 24 in the four-qubit amplitudes takes a more instructive form than the previously published expressions available in the literature. Finally, in order to understand two-, three- and four-qubit entanglement in geometric terms we propose a unified setting based on \mathbf{CP}^3 furnished with a fixed quadric.

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1. Introduction

Recently, the problem of characterizing n -qubit entanglement classes has generated considerable interest. This problem was raised within the context of quantum information theory regarding the physical phenomenon of entanglement as a resource. In order to exploit the capabilities encoded in this resource for different tasks of quantum information processing we have to somehow measure it. During the past few years a number of useful entanglement measures for pure states have been found [1–9]. Classifications up to four qubits have appeared [10–12] and the interesting geometric structures associated with entangled pure states have been noted [9, 12–18].

In our previous set of papers [9, 16, 17] using some results from twistor theory we initiated an approach for understanding n -qubit entanglement in geometric terms. We have shown that this problem can be completely solved for three qubits, and we obtained partial results for a special subclass characterized by n -qubit entanglement monotones. The aim of

the present paper is to add some interesting new results on the geometry of four-qubit SLOCC (stochastic local operations and classical communication [19]) invariants. Such invariants have been introduced in [5], where the Hilbert series of the algebra of invariants has been found. This result enabled the authors to construct a complete set of four algebraically independent invariants of degrees 2, 4, 4, 6, respectively in the complex coefficients characterizing the four-qubit entangled pure state. The values of these invariants on the SLOCC orbits of [11] were also given. Moreover, an explicit formula for the hyperdeterminant of degree 24 was also obtained. The authors of [5] have conjectured that some of the invariants might have a geometric meaning. In this paper we show that this is indeed the case.

In section 2 we present a new set of four invariants, by replacing two from those of [5]. In section 3 we clarify the geometric meaning of these invariants in terms of the geometry of \mathbf{CP}^3 , the complex projective space. In section 4 we show that using our new set of invariants the expression for the 24th-order hyperdeterminant takes a more instructive form than the one that can be found in [5]. Moreover, it turns out that the entanglement classes invariant under SLOCC transformations take a very simple form related to the elementary symmetric polynomials in four complex variables. The magnitudes of our invariants turn out to be entanglement monotones that fit nicely into the geometric set of n -qubit entanglement monotones related to Grassmannians of l -planes in \mathbf{C}^L , with $L = 2^n, l \leq L$ found recently. Finally, our conclusions and some comments are left for section 5.

2. Invariants

Let us write an arbitrary four-qubit state in the form

$$|\Psi\rangle = \sum_{i,j,k,l=0}^1 Z_{ijkl} |ijkl\rangle \in \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2, \tag{1}$$

where $|ijkl\rangle = |i\rangle_1 \otimes |j\rangle_2 \otimes |k\rangle_3 \otimes |l\rangle_4$. Following [5] we introduce decimal notation for $Z_{ijkl} \equiv Z_r$, where $r = 8i + 4j + 2k + l$, and the matrices

$$\mathcal{L} = \begin{Bmatrix} Z_0 & Z_4 & Z_8 & Z_{12} \\ Z_1 & Z_5 & Z_9 & Z_{13} \\ Z_2 & Z_6 & Z_{10} & Z_{14} \\ Z_3 & Z_7 & Z_{11} & Z_{15} \end{Bmatrix} \equiv (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \tag{2}$$

$$\mathcal{M} = \begin{pmatrix} Z_0 & Z_2 & Z_8 & Z_{10} \\ Z_1 & Z_3 & Z_9 & Z_{11} \\ Z_4 & Z_6 & Z_{12} & Z_{14} \\ Z_5 & Z_7 & Z_{13} & Z_{15} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{A}^T & \mathcal{C}^T \\ \mathcal{B}^T & \mathcal{D}^T \end{pmatrix} \tag{3}$$

$$\mathcal{N} = \begin{pmatrix} Z_0 & Z_1 & Z_8 & Z_9 \\ Z_2 & Z_3 & Z_{10} & Z_{11} \\ Z_4 & Z_5 & Z_{12} & Z_{13} \\ Z_6 & Z_7 & Z_{14} & Z_{15} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{B} & \mathcal{D} \end{pmatrix}. \tag{4}$$

Here $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbf{C}^4$ are considered as four column vectors and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are 2×2 matrices with T referring to transposition. We wish to describe the geometry of four-qubit entanglement in terms of the four vectors $A_\alpha, B_\beta, C_\gamma$ and D_δ living in \mathbf{C}^4 , where $\alpha, \beta, \gamma, \delta = 0, 1, 2, 3$. Hence we regard the matrix \mathcal{L} as fundamental. The matrices \mathcal{M} and \mathcal{N} will be used later.

We are interested in studying a subset of polynomials in the complex numbers $Z_r, r = 0, \dots, 15$, that are invariant under the SLOCC group of stochastic local operations and classical communication, i.e. $SL(2, \mathbf{C})^{\otimes 4}$. Such transformations are of the form

$$|\Psi\rangle \mapsto (S_1 \otimes S_2 \otimes S_3 \otimes S_4)|\Psi\rangle, \tag{5}$$

where $S_m \in SL(2, \mathbf{C}), m = 1, 2, 3, 4$, and $|\Psi\rangle$ takes the form (1) with the indices of S_m referring to the label of \mathbf{C}^2 in the tensor product they are acting on.

In order to define the SLOCC invariants we will introduce two extra structures on \mathbf{C}^4 . The first one is a bilinear form $g : \mathbf{C}^4 \times \mathbf{C}^4 \rightarrow \mathbf{C}$ such that for two vectors $\mathbf{A}, \mathbf{B} \in \mathbf{C}^4$ we have

$$(\mathbf{A}, \mathbf{B}) \mapsto g(\mathbf{A}, \mathbf{B}) \equiv \mathbf{A} \cdot \mathbf{B} = g_{\alpha\beta} A^\alpha B^\beta = A_\alpha B^\alpha, \tag{6}$$

where

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{7}$$

$\alpha, \beta = 0, 1, 2, 3$, and summation for repeated indices is understood. Note that the matrix of our symmetric bilinear form can be written in a tensor product form $g = \varepsilon \otimes \varepsilon$, where ε is invariant under the group $SL(2, \mathbf{C})$, i.e. we have $S\varepsilon S^T = \varepsilon$ with $S \in SL(2, \mathbf{C})$. Moreover, since

$$\begin{aligned} A_\alpha &= Z_{00kl}, & B_\beta &= Z_{01kl}, & C_\gamma &= Z_{10kl}, \\ D_\delta &= Z_{11kl}, & \alpha, \beta, \gamma, \delta &= 0, 1, 2, 3, & k, l &= 0, 1, \end{aligned} \tag{8}$$

quantities involving this symmetric bilinear form are automatically invariant with respect to $SL(2, \mathbf{C}) \otimes SL(2, \mathbf{C})$ transformations of third and fourth qubit, i.e. those of the form $I \otimes I \otimes S_3 \otimes S_4$. Since $SL(2, \mathbf{C}) \otimes SL(2, \mathbf{C})/\mathbf{Z}_2 \simeq SO(4, \mathbf{C})$ it follows that greek indices like α, β etc can also be regarded as vector indices under $SO(4, \mathbf{C})$. This conversion of complex four vectors into complex 2×2 matrices has already been used elsewhere to connect the results of twistor theory to the geometry of entanglement [16, 17]. Hence, the columns of the matrix \mathcal{L} of (2) transform as vectors under transformations of the form $I \otimes I \otimes S_3 \otimes S_4$ and as the 00, 01, 10 and 11 components of a tensor under those with the form $S_1 \otimes S_2 \otimes I \otimes I$.

The second structure, as we will see, is related to the notion of duality in \mathbf{CP}^3 . For the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and \mathbf{D} let us introduce their *duals* as

$$\begin{aligned} a_\alpha &\equiv -\epsilon_{\alpha\beta\gamma\delta} B^\beta C^\gamma D^\delta, & b_\beta &= \epsilon_{\alpha\beta\gamma\delta} A^\alpha C^\gamma D^\delta, \\ c_\gamma &\equiv \epsilon_{\alpha\beta\gamma\delta} A^\alpha B^\beta D^\delta, & d_\delta &\equiv -\epsilon_{\alpha\beta\gamma\delta} A^\alpha B^\beta C^\gamma. \end{aligned} \tag{9}$$

Clearly, these quantities transform as vectors under transformations of the form $I \otimes I \otimes S_3 \otimes S_4$ and a straightforward calculation shows that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and \mathbf{d} behave under those of the form $S_1 \otimes S_2 \otimes I \otimes I$ as the 11, 10, 01 and 00 components of a tensor, respectively.

Let us now introduce the notation $\mathbf{A} \wedge \mathbf{B}$ (a bivector) corresponding to the antisymmetric matrix $A_\alpha B_\beta - A_\beta B_\alpha$ and the one

$$L \equiv \text{Det}\mathcal{L} = \epsilon^{\alpha\beta\gamma\delta} A_\alpha B_\beta C_\gamma D_\delta. \tag{10}$$

Now the SLOCC invariants we wish to propose are

$$I_1 = \frac{1}{2}(\mathbf{A} \cdot \mathbf{D} - \mathbf{B} \cdot \mathbf{C}) \tag{11}$$

$$I_2 = \frac{1}{6}((\mathbf{A} \wedge \mathbf{B}) \cdot (\mathbf{C} \wedge \mathbf{D}) + (\mathbf{A} \wedge \mathbf{C}) \cdot (\mathbf{B} \wedge \mathbf{D}) - \frac{1}{2}(\mathbf{A} \wedge \mathbf{D})^2 - \frac{1}{2}(\mathbf{B} \wedge \mathbf{C})^2) \tag{12}$$

$$I_3 = \frac{1}{2}(\mathbf{a} \cdot \mathbf{d} - \mathbf{b} \cdot \mathbf{c}) \tag{13}$$

$$I_4 = L. \tag{14}$$

Here quantities such as $(\mathbf{A} \wedge \mathbf{B}) \cdot (\mathbf{C} \wedge \mathbf{D})$ are defined as

$$\begin{aligned}
 (\mathbf{A} \wedge \mathbf{B}) \cdot (\mathbf{C} \wedge \mathbf{D}) &= (A_\alpha B_\beta - A_\beta B_\alpha)(C^\alpha D^\beta - C^\beta D^\alpha) \\
 &= 2((\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})).
 \end{aligned}
 \tag{15}$$

The first of our invariants I_1 takes the form

$$I_1 = \frac{1}{2}H = \frac{1}{2}(Z_0Z_{15} - Z_1Z_{14} - Z_2Z_{13} + Z_3Z_{12} - Z_4Z_{11} + Z_5Z_{10} + Z_6Z_9 - Z_7Z_8), \tag{16}$$

showing that $I_1 = \frac{1}{2}H$, where H is one of the basic invariants of [5]. It is just a special case of the n -tangle with n even introduced earlier by Wong and Christensen [3]. Reverting to binary notation it is easy to show that I_1 is also a permutation invariant [3, 5, 9].

Our last invariant I_4 is just $L = \text{Det}\mathcal{L}$, an invariant also introduced by the authors of [5]. These authors have also introduced two more invariants of orders 4 and 6 denoted by M and D , respectively (the first of them being just *minus* the determinant of our matrix \mathcal{M}). They have shown after obtaining the Hilbert series that the invariants H, L, M and D are algebraically independent and complete. Here, instead of the invariants M and D we prefer the new ones I_2 and I_3 . As we will see the set (I_1, I_2, I_3, I_4) is of geometrical significance. Moreover, it turns out that the values of these invariants on the generic SLOCC orbit [11] of four-qubit entangled states are just the elementary symmetric polynomials in four complex variables. We will also show that in terms of this new set of invariants the explicit formula for the hyperdeterminant of degree 24 takes a more instructive form than the corresponding one of [5].

3. The geometric meaning of four-qubit invariants

3.1. The invariant I_2

Let us now explain the structure of I_2 . In order to do this we introduce another \mathbf{C}^4 corresponding to the four-vector structure also present in the *first two* indices of Z_{ijkl} . Converting the first two spinor indices to the vector ones labelled by $\mu = 0, 1, 2, 3$, what we obtain is a ‘vector-valued’ four-vector $Z_{\mu\alpha} = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})^T$. (Alternatively, regarding $Z_{\mu\alpha}$ as a 4×4 matrix we obtain the matrix \mathcal{L} of equation (2).) Let us now also supply this new copy of \mathbf{C}^4 with the bilinear form g known from equation (6) with matrix $g_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$. Define now the second exterior power of a matrix as the map

$$\bigwedge^2 : \mathbf{C}^{n \times n} \rightarrow \mathbf{C}^{\binom{n}{2} \times \binom{n}{2}}, \tag{17}$$

which takes an $M_{\mu\nu} \in \mathbf{C}^{n \times n}$, $0 \leq \mu, \nu \leq n - 1$, to

$$M^{(2)} \equiv \left(\bigwedge^2 M \right)_{IJ} \equiv M_{\mu_1\nu_1}M_{\mu_2\nu_2} - M_{\mu_1\nu_2}M_{\mu_2\nu_1}, \tag{18}$$

where $I = \{\mu_1, \mu_2\}$ with $0 \leq \mu_1 < \mu_2 \leq n - 1$ and $J = \{\nu_1, \nu_2\}$ with $0 \leq \nu_1 < \nu_2 \leq n - 1$. For the 4×4 matrix $g_{\mu\nu}$ of our bilinear form g we have

$$G_{IJ} \equiv g_{IJ}^{(2)} = g^{(2)IJ} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{19}$$

where $I, J = 01, 02, 03, 12, 13, 23$.

Let us now introduce the ‘bivector-valued’ 4×4 antisymmetric Plücker matrix

$$P_{\mu\nu} = \begin{pmatrix} 0 & \mathbf{A} \wedge \mathbf{B} & \mathbf{A} \wedge \mathbf{C} & \mathbf{A} \wedge \mathbf{D} \\ -\mathbf{A} \wedge \mathbf{B} & 0 & \mathbf{B} \wedge \mathbf{C} & \mathbf{B} \wedge \mathbf{D} \\ -\mathbf{A} \wedge \mathbf{C} & -\mathbf{B} \wedge \mathbf{C} & 0 & \mathbf{C} \wedge \mathbf{D} \\ -\mathbf{A} \wedge \mathbf{D} & -\mathbf{B} \wedge \mathbf{D} & -\mathbf{C} \wedge \mathbf{D} & 0 \end{pmatrix}. \tag{20}$$

Note that since the elements of $P_{\mu\nu}$ are separable bivectors it has the index structure $P_{\mu\nu\alpha\beta}$. Explicitly, we have

$$P_{\mu\nu\alpha\beta} = Z_{\mu\alpha}Z_{\nu\beta} - Z_{\mu\beta}Z_{\nu\alpha}. \tag{21}$$

They are the Plücker coordinates of six lines in \mathbf{CP}^3 labelled by the subscripts $\mu\nu = 01, 02, 03, 12, 13, 23$. Alternatively, one can introduce the ‘bivector-valued’ vector \mathbf{P} with six components $P_I, I = 01, 02, 03, 12, 13, 23$, with the same geometric meaning:

$$P_I = (\mathbf{A} \wedge \mathbf{B}, \mathbf{A} \wedge \mathbf{C}, \mathbf{A} \wedge \mathbf{D}, \mathbf{B} \wedge \mathbf{C}, \mathbf{B} \wedge \mathbf{D}, \mathbf{C} \wedge \mathbf{D})^T. \tag{22}$$

In this notation the invariant I_2 of equation (12) can be written in the form

$$I_2 = \frac{1}{12} G^{IJ} P_I \cdot P_J = \frac{1}{6} P^{\mu\nu} \cdot P_{\mu\nu} = \frac{1}{6} P^{\mu\nu\alpha\beta} P_{\mu\nu\alpha\beta}. \tag{23}$$

Note that the elements of the Plücker matrix being now separable bivectors are multiplied together according to the (15) law bringing in the dot product. Since I_2 contains contractions with respect to four $SL(2, \mathbf{C}) \otimes SL(2, \mathbf{C})$ invariant matrices g (two of them operate on the first two and the other two on the last two qubits), it is automatically an $SL(2, \mathbf{C})^{\otimes 4}$ invariant.

From the form of the Plücker matrix equation (20) it is also clear that it encapsulates information concerning four-qubit entanglement in terms of *six planes* in \mathbf{C}^4 or alternatively *six lines* in \mathbf{CP}^3 . Hence, we managed to identify the SLOCC invariant I_2 as a *line invariant* in complex projective space.

Let us now clarify the relationship of our invariant I_2 with those of [5]. There also the invariants $L = \text{Det}\mathcal{L}, M = \text{Det}\mathcal{M}$ and $N = \text{Det}\mathcal{N}$ were defined where the 4×4 matrices are given by equations (2)–(4). Note that our convention for M differs in sign from the one adopted in [5]. It can be shown that $M = L + N$ as can be verified by calculating the determinants of the matrices in equations (3) and (4) containing 2×2 blocks. Now a straightforward calculation shows that in terms of the algebraically independent invariants $H = 2I_1, L = I_4$ and M preferred by [5] we have the relation

$$6I_2 = H^2 + 2L - 4M. \tag{24}$$

3.2. The invariant I_3

In order to understand the geometric meaning of the invariant I_3 of equation (13) we have to recall some results from projective geometry. A *plane* in \mathbf{CP}^3 consists of a set of points with homogeneous coordinates $X^\alpha, \alpha = 0, 1, 2, 3$, which satisfy a single linear equation of the form

$$a_\alpha X^\alpha = 0, \tag{25}$$

where the complex numbers a_α are called the *coordinates of the plane*. Clearly, a_α and λa_α , with $\lambda \neq 0$, determine the same plane in \mathbf{CP}^3 , so the set of planes in \mathbf{CP}^3 is itself a \mathbf{CP}^3 called the *dual projective space*. A plane in \mathbf{CP}^3 is a \mathbf{CP}^2 . There is a unique plane containing three general points in \mathbf{CP}^3 . If B_β, C_γ and D_δ are the three general points, then there is a unique solution up to proportionality of the three equations

$$a_\alpha B^\alpha = a_\alpha C^\alpha = a_\alpha D^\alpha = 0 \tag{26}$$

given by the first of equation (9). Now we see that the four vectors $a_\alpha, b_\beta, c_\gamma$ and d_δ defined in equation (9) are the (dual Plücker) coordinates of four planes in \mathbf{CP}^3 . They are defined by the four points $(\mathbf{B}, \mathbf{C}, \mathbf{D}), (\mathbf{A}, \mathbf{C}, \mathbf{D}), (\mathbf{A}, \mathbf{B}, \mathbf{D})$ and $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, respectively. Alternatively, for these planes we can use the Plücker coordinates (separable trivectors) $\mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D}, \mathbf{A} \wedge \mathbf{C} \wedge \mathbf{D}, \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{D}, \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}$, where for example

$$(\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C})_{\alpha\beta\gamma} = A_\alpha B_\beta C_\gamma + A_\gamma B_\alpha C_\beta + A_\beta B_\gamma C_\alpha - A_\beta B_\alpha C_\gamma - A_\gamma B_\beta C_\alpha - A_\alpha B_\gamma C_\beta \equiv 3! A_{[\alpha} B_\beta C_{\gamma]}. \tag{27}$$

The set of points common to two planes is a *line* in \mathbf{CP}^3 . A line is given by the points satisfying two linear equations of the form

$$a_\alpha X^\alpha = b_\alpha X^\alpha = 0. \tag{28}$$

A sufficient and necessary condition for these equations to hold is

$$a_{[\alpha} b_{\beta]} X^\beta = 0. \tag{29}$$

Hence, in order to characterize the line (a \mathbf{CP}^1) defined by this equation we can either use the *dual Plücker coordinates* (a separable bivector)

$$(\mathbf{a} \wedge \mathbf{b})_{\alpha\beta} = a_\alpha b_\beta - a_\beta b_\alpha \equiv 2! a_{[\alpha} b_{\beta]} \tag{30}$$

or using equation (9) the *Plücker coordinates* (another separable bivector)

$$(\mathbf{C} \wedge \mathbf{D})_{\gamma\delta} = C_\gamma D_\delta - C_\delta D_\gamma \equiv 2! C_{[\gamma} D_{\delta]}. \tag{31}$$

This example shows that the planes with the Plücker coordinates $\mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D}$ and $\mathbf{A} \wedge \mathbf{C} \wedge \mathbf{D}$ intersect in the projective line given by the Plücker coordinates $\mathbf{C} \wedge \mathbf{D}$.

Now clearly I_3 of equation (13) is an $SL(2, \mathbf{C})^{\otimes 4}$ invariant. Indeed, the dual Plücker coordinates $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and \mathbf{d} are transforming as the 11, 01, 10 and 00 components of a tensor under transformations of the form $S_1 \otimes S_2 \otimes I \otimes I$; hence, the combination $\mathbf{a} \cdot \mathbf{d} - \mathbf{b} \cdot \mathbf{c}$ is an invariant with respect to such transformations due to the ‘determinant-like’ structure. Moreover, this quantity is also invariant under transformations of the form $I \otimes I \otimes S_3 \otimes S_4$ due to the occurrence of the $SL(2, \mathbf{C})^{\otimes 2}$ invariant dot product of equation (6). Note that I_3 has the same structure as I_1 . This exemplifies a general pattern: suppose we have an invariant (I_1); then find a set of *covariants* (i.e. $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and \mathbf{d}) to construct a new invariant (I_3) by exploiting the existing structure of the original invariant (I_1). It is clear that I_1 describes constellations of *points* and I_3 describes *planes* in \mathbf{CP}^3 . It is important to realize, however, that unlike I_3 the invariant I_1 is also a permutation invariant.

Let us also express our invariant in terms of the Plücker coordinates $\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}$ etc as

$$I_3 = \frac{1}{12} ((\mathbf{A} \wedge \mathbf{C} \wedge \mathbf{D}) \cdot (\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{D}) - (\mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D}) \cdot (\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C})). \tag{32}$$

Alternatively, like in equation (20) one can define a third-order totally antisymmetric Plücker tensor $P_{\mu\nu\rho}$ with trivectors as elements. The four independent elements of $P_{\mu\nu\rho}$ are $P_{012} = \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}, P_{123} = \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D}, P_{023} = \mathbf{A} \wedge \mathbf{C} \wedge \mathbf{D}$ and $P_{013} = \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{D}$. Note that these quantities have the index structure e.g. $P_{012\alpha\beta\gamma} = (\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C})_{\alpha\beta\gamma}$ where the definition of equation (27) holds. Using this notation we have

$$I_3 = \frac{1}{12^2} P^{\mu\nu\rho} \cdot P_{\mu\nu\rho} = \frac{1}{12^2} P^{\mu\nu\rho\alpha\beta\gamma} P_{\mu\nu\rho\alpha\beta\gamma}, \tag{33}$$

an expression to be compared with equation (23) obtained for our line invariant.

Finally, let us relate our invariant I_3 to the sixth-order ones of [5]. Define the quadrilinear form

$$Z(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}) = \sum_{i,j,k,l=0}^1 Z_{ijkl} x_i y_j z_k t_l. \tag{34}$$

Using this for each pair of variables one then defines the covariants (now we define it for the pair xy)

$$b_{xy}(\mathbf{x}, \mathbf{z}) = \text{Det} \left(\frac{\partial^2 Z}{\partial z_i \partial t_j} \right). \tag{35}$$

Now let us reinterpret these biquadratic forms as bilinear forms on $S^2\mathbf{C}$ (the symmetric part of $\mathbf{C}^2 \otimes \mathbf{C}^2$) as

$$b_{xy}(\mathbf{x}, \mathbf{y}) = (x_0^2, x_0x_1, x_1^2) B_{xy} \begin{pmatrix} y_0^2 \\ y_0y_1 \\ y_1^2 \end{pmatrix}, \tag{36}$$

i.e. B_{xy} is a 3×3 matrix. Then following [5] we define

$$D_{uv} = \text{Det}(B_{uv}). \tag{37}$$

Hence, we have six sextic invariants $D_{xy}, D_{zt}, D_{xz}, D_{yt}, D_{xt}$ and D_{yz} . According to [5] only four of them are independent due to the relations $D_{xy} = D_{zt}, D_{xz} = D_{yt}$ and $D_{xt} = D_{yz}$. Now a straightforward calculation shows that

$$I_3 = \frac{1}{2}(D_{xz} + D_{xt}). \tag{38}$$

In [5] the authors used the invariant $D \equiv D_{xt}$ as a fundamental one satisfying the relation $D_{xz} - D_{xt} = HL$; hence we can write

$$I_3 = D + \frac{1}{2}HL. \tag{39}$$

By virtue of equations (14), (16), (24) and (39) the relationship between our set of invariants (I_1, I_2, I_3, I_4) and those (H, M, L, D) used in [5] is established.

3.3. The invariants I_4 and I_2

The meaning of I_4 is clear. $I_4 = \text{Det}\mathcal{L}$ vanishes when the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and \mathbf{D} are linearly dependent. Moreover, since I_4 and I_2 are both of fourth order let us now explore the relationship between them.

Let us label the six lines as in equation (20). Hence, for example, P_{01} is the line corresponding to the separable bivector $\mathbf{A} \wedge \mathbf{B}$. More precisely, this object has the index structure $(P_{01})_{\alpha\beta} = (\mathbf{A} \wedge \mathbf{B})_{\alpha\beta} = A_\alpha B_\beta - A_\beta B_\alpha$. In this notation

$$L = \frac{1}{4}\varepsilon^{\alpha\beta\gamma\delta}(P_{01})_{\alpha\beta}(P_{23})_{\gamma\delta}. \tag{40}$$

We can regard this expression as a symmetric bilinear form in the six Plücker coordinates of the two lines $\mathbf{A} \wedge \mathbf{B}$ and $\mathbf{C} \wedge \mathbf{D}$. Let us denote this bilinear form by $\langle \cdot, \cdot \rangle$; hence we have

$$\langle \cdot, \cdot \rangle : \bigwedge^2 \mathbf{C}^4 \otimes \bigwedge^2 \mathbf{C}^4 \rightarrow \mathbf{C}, \quad (P_{\mu\nu}, P_{\rho\sigma}) \mapsto \langle P_{\mu\nu}, P_{\rho\sigma} \rangle. \tag{41}$$

Defining the dual of a bivector as

$${}^*P_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}P^{\gamma\delta}, \tag{42}$$

it is easy to show that

$$\langle P_{\mu\nu}, P_{\rho\sigma} \rangle = \langle {}^*P_{\mu\nu}, {}^*P_{\rho\sigma} \rangle. \tag{43}$$

In this notation the equation $\langle P_{\mu\nu}, P_{\nu\rho} \rangle = 0$ expresses the fact that the planes described by the separable bivectors $P_{\mu\nu}$ and $P_{\nu\rho}$ in \mathbf{C}^4 have a line in common. In the \mathbf{CP}^3 picture this is

equivalent to the fact that the corresponding *lines* in \mathbf{CP}^3 intersect *at a point*. Hence, we can look at L as a line invariant too; moreover we have the obvious relations

$$I_4 = \langle P_{01}, P_{23} \rangle = \langle P_{02}, P_{31} \rangle = \langle P_{03}, P_{12} \rangle = \langle *P_{01}, *P_{23} \rangle = \langle *P_{02}, *P_{31} \rangle = \langle *P_{03}, *P_{12} \rangle. \tag{44}$$

Let us now look at quantities like $\langle P_{\mu\nu}, *P_{\rho\sigma} \rangle$! We have for example

$$\langle P_{01}, *P_{23} \rangle = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) = \frac{1}{2}(\mathbf{A} \wedge \mathbf{B}) \cdot (\mathbf{C} \wedge \mathbf{D}). \tag{45}$$

Since $\langle P_{\mu\nu}, *P_{\rho\sigma} \rangle = \langle *P_{\mu\nu}, P_{\rho\sigma} \rangle$ one can now write the invariant $6I_2$ in the form

$$6I_2 = \langle P_{01}, *P_{23} \rangle + \langle *P_{01}, P_{23} \rangle + \langle P_{02}, *P_{13} \rangle + \langle *P_{02}, P_{13} \rangle - \langle P_{03}, *P_{03} \rangle - \langle P_{12}, *P_{12} \rangle. \tag{46}$$

Using equation (43) let us calculate $6(I_2 \pm I_4)$. We get

$$6(I_2 \pm I_4) = \pm \langle P_{01} \pm *P_{01}, P_{23} \pm *P_{23} \rangle \pm \langle P_{02} \mp *P_{02}, P_{31} \mp *P_{31} \rangle - \langle P_{12} \mp *P_{03}, *P_{12} \mp P_{03} \rangle. \tag{47}$$

As we will see these invariants will occur in the expression for the hyperdeterminant. The geometric meaning of these invariants is connected to the intersection properties of the *self-dual* ($*P = P$) or *anti-self-dual* ($*P = -P$) parts of the planes in \mathbf{C}^4 (or alternatively of lines in \mathbf{CP}^3). For example for P_{01} self-dual, P_{31} anti-self-dual and P_{12} identical to the dual line of P_{30} (an equivalent condition for this is $L(\mathbf{B} \wedge \mathbf{C}) = -\mathbf{b} \wedge \mathbf{c}$) the invariant $I_4 - I_2$ vanishes. It is easy to check that the invariant U occurring in [5] can be related to one of these invariants as

$$U \equiv H^2 - 4(L + M) = 6(I_4 - I_2). \tag{48}$$

The fact that (among others) this invariant might have a geometric meaning was raised in [5].

4. The hyperdeterminant

Let us now consider the hyperdeterminant D_4 for the four-qubit system. As is well known for two-qubit systems the determinant $D_2 = Z_{00}Z_{11} - Z_{01}Z_{10}$ is related to the *concurrence* [2] as $\mathcal{C} = 2|D_2|$ characterizing two-qubit entanglement. Similarly, for three qubits the basic quantity is the three-tangle [2] $\tau = 4|D_3|$, which is related to the hyperdeterminant D_3 of a $2 \times 2 \times 2$ tensor formed from the eight complex amplitudes Z_{ijk} . D_3 is an irreducible polynomial in the eight amplitudes which is the sum of 12 terms of degree 4. For the explicit expression of D_3 see e.g. the book of Gelfand *et al* [20]. It is known that the next item in the line namely the hyperdeterminant D_4 of the form $2 \times 2 \times 2 \times 2$ is a polynomial of degree 24 in the 16 amplitudes Z_{ijkl} which has 2 894 276 terms [21]. An expression in terms of the fundamental invariants (H, L, M, D) was given in [5]. Here we are interested in the explicit form of D_4 , the hyperdeterminant of the $2 \times 2 \times 2 \times 2$ tensor Z_{ijkl} , based on the special invariants (I_1, I_2, I_3, I_4) we have found in our \mathbf{CP}^3 picture.

As is well known [20] the hyperdeterminant D_4 is the unique irreducible polynomial in the 16 unknowns Z_{ijkl} that vanishes whenever the system of equations

$$F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = \frac{\partial F}{\partial t} = 0, \tag{49}$$

where

$$F = Z_{0000} + Z_{0001}t + Z_{0010}z + Z_{0100}y + Z_{1000}x + Z_{0011}zt + Z_{0101}yt + \dots + Z_{1110}xyz + Z_{1111}xyzt, \tag{50}$$

has a solution (x_0, y_0, z_0, t_0) in \mathbf{C}^4 .

Using the method of Schläfli according to theorem 14.4.1 and corollary 14.2.10 of [20], D_4 coincides with the discriminant Δ of $D_3(Z_{0jkl} + \lambda Z_{1jkl})$ considered as a polynomial in λ divided by 256. This method has already been used to obtain a much simpler form for D_3 of geometric meaning [16]. For D_4 a method equivalent to this has been applied with the result [5]

$$256D_4 = S^3 - 27T^2, \tag{51}$$

where

$$12S = U^2 - 2V, \quad 216T = U^3 - 3UV + 216D^2, \tag{52}$$

with

$$U = H^2 - 4(L + M), \quad V = 12(HD + 2LM). \tag{53}$$

Let us now express D_4 in terms of our invariants (I_1, I_2, I_3, I_4) of geometric significance. Using relations (14), (16), (24) and (39) in equations (51)–(53) we obtain the result

$$S = (I_4^2 - I_2^2) + 4(I_2^2 - I_1I_3), \quad T = (I_4^2 - I_2^2)(I_1^2 - I_2) + (I_3 - I_1I_2)^2. \tag{54}$$

In this form it is obvious that the combined invariants $I_4 \pm I_2, I_1^2 - I_2$ of fourth, $(I_3 - I_1I_2)$ of sixth and $I_2^2 - I_1I_3$ of eighth order should play a basic geometric role. We have already clarified the geometric meaning of the first two invariants. They are related to self-duality and anti-self-duality of the corresponding lines in \mathbf{CP}^3 . One of these invariants $I_4 - I_2$ is just $\frac{1}{6}U$, also used in [5]. In our form of D_4 we prefer to also use the dual combination $I_4 + I_2$. For the time being we do not know any geometrical interpretation of the other combinations. Intuitively, it is clear that the invariant $I_1^2 - I_2$ should play a similar role to the other fourth-order invariants. Indeed, we have chosen the third and fourth qubits to play a special role. (An equivalent picture arises when a special role is assigned to the first and the second qubit.) \mathbf{C}^4 defined by them is equipped with the bilinear form of equation (6). The null vectors (i.e. those satisfying $\mathbf{X} \cdot \mathbf{X} = 0$) describe a quadric embedded in \mathbf{CP}^3 which is isomorphic to $\mathbf{CP}^1 \times \mathbf{CP}^1$, i.e. it is ruled by two families of projective lines which can be shown to be self-dual or anti-self-dual, respectively [23]. Projective lines lying entirely inside a fixed quadric are called isotropic lines. Had we chosen the two qubits playing a special role differently the notion of self- or anti-self-duality of isotropic lines would have been defined with respect to a different quadric. In this picture we conjecture that the invariants $I_4 - I_2$ and $I_1^2 - I_2$ would play a dual role. Since altogether we have three inequivalent choices, therefore, we can conclude that the fourth-order invariants are related to the notion of duality of isotropic lines with respect to a fixed quadric in \mathbf{CP}^3 . It would be interesting to find a geometric interpretation for the remaining invariants too.

Let us now consider another interesting property of D_4 expressed in terms of our invariants (I_1, I_2, I_3, I_4) . As is well known the discriminant Δ of the polynomial $e_4w^4 + e_3w^3 + e_2w^2 + e_1w + e_0$ is given by the expression (see [20], equation (1.35) on p 405)

$$\begin{aligned} \Delta(e_4w^4 + e_3w^3 + e_2w^2 + e_1w + e_0) = & 256e_0^3e_4^2 - 192e_0^2e_1e_3e_4^2 - 128e_0^2e_2^2e_4^2 \\ & + 144e_0^2e_2e_3^2e_4 - 27e_0^2e_3^4 + 144e_0e_1^2e_2e_4^2 - 6e_0e_1^2e_3^2e_4 - 80e_0e_1e_2^2e_3e_4 \\ & + 18e_0e_1e_2e_3^3 + 16e_0e_2^4e_4 - 4e_0e_2^3e_3^3 - 27e_1^4e_4^2 + 18e_1^3e_2e_3e_4 - 4e_1^3e_3^3 \\ & - 4e_1^2e_2^3e_4 + e_1^2e_2^2e_3^4. \end{aligned} \tag{55}$$

Let us now consider the polynomial of the special form

$$p[I_1, I_2, I_3, I_4; w] \equiv w^4 - (4I_1)w^3 + (6I_2)w^2 - (4I_3)w + I_4^2. \tag{56}$$

Then a straightforward calculation shows that

$$256D_4 = \Delta(p[I_1, I_2, I_3, I_4; w]). \quad (57)$$

Note that the polynomial p is not directly related to the one arising from the method of Schläfli. In this case one obtains

$$D_3(Z_{0jkl} + \lambda Z_{1jkl}) = h_4\lambda^4 + h_3\lambda^3 + h_2\lambda^2 + h_1\lambda + h_0, \quad (58)$$

where unlike those e_s , the coefficients h_s , $s = 0, 1, \dots, 4$, are fourth-order polynomials of Z_{ijkl} that are *not invariant* with respect to the full group $SL(2, \mathbf{C})^{\otimes 4}$. However, the discriminant of this polynomial again gives $256D_4$, which is already an invariant with respect to the full group of SLOCC transformations.

In order to illustrate the advantages of using our invariants (I_1, I_2, I_3, I_4) let us now calculate their values on the generic SLOCC class. A generic pure state of four qubits can always be transformed to the form [11]

$$\begin{aligned} |G_{abcd}\rangle = & \frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle + |1100\rangle) \\ & + \frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle), \end{aligned} \quad (59)$$

where a, b, c, d are the complex numbers. For this state the reduced density matrices obtained by tracing out all but one party are proportional to the identity. This is the state with maximal four-partite entanglement. Another interesting property of this state is that it does not contain true three-partite entanglement [11]. A straightforward calculation shows that the values of our invariants (I_1, I_2, I_3, I_4) occurring for the state $|G_{abcd}\rangle$ representing the generic SLOCC class are

$$I_1 = \frac{1}{4}[a^2 + b^2 + c^2 + d^2], \quad (60)$$

$$I_2 = \frac{1}{6}[(ab)^2 + (ac)^2 + (ad)^2 + (bc)^2 + (bd)^2 + (cd)^2],$$

$$I_3 = \frac{1}{4}[(abc)^2 + (abd)^2 + (acd)^2 + (bcd)^2], \quad I_4 = abcd, \quad (61)$$

hence, the values of the invariants $(4I_1, 6I_2, 4I_3, I_4^2)$ occurring in the polynomial equation (56) are given in terms of the elementary symmetric polynomials in the variables $(x_1, x_2, x_3, x_4) = (a^2, b^2, c^2, d^2)$. From this and equation (57) it immediately follows that the value of the hyperdeterminant on the SLOCC orbit represented by the state $|G_{abcd}\rangle$ is

$$D_4 = \frac{1}{256} \prod_{i < j} (x_i - x_j)^2 = \frac{1}{256} V(a^2, b^2, c^2, d^2)^2, \quad (62)$$

in accordance with [5], where V is the Vandermonde determinant. For the other SLOCC classes and the values of the invariants (H, L, M, D) see [5]. It is straightforward to give the alternative values of (I_1, I_2, I_3, I_4) on these classes.

5. Comments and conclusions

In this paper we have considered some aspects of the problem of understanding four-qubit entanglement in geometric terms. We have replaced two from the set containing four algebraically independent invariants of [5] by the new ones. In this way all four invariants have a simple geometric meaning. I_1 is based on four 0-planes (points), I_2 on six 1-planes (lines), I_3 on four 2-planes (planes) and finally I_4 on a single 3-plane in \mathbf{CP}^3 . According to theorem 2 of [24] suitable powers of the magnitudes of these invariants can be used as entanglement monotones characterizing four-qubit entanglement. Moreover, for an arbitrary four-qubit

state after calculating the set of invariants (I_1, I_2, I_3, I_4) and the value of the hyperdeterminant D_4 , for $D_4 \neq 0$, we obtain four different roots of the fourth-order equation (56). These roots are just the complex numbers (a^2, b^2, c^2, d^2) . Their square roots produce the values $(\pm a, \pm b, \pm c, \pm d)$ appearing in the canonical form G_{abcd} . This shows that the study of the degenerate cases of multiple roots of equation (56) arising for $D_4 = 0$ could be useful for obtaining the parameters of the canonical forms [11] from the values of the basic invariants. This process is similar to the spirit of the one found for the three-qubit case [26]. There by calculating the values of the independent $SU(2)^{\otimes 3}$ invariants the canonical form of an arbitrary three-qubit state was found.

We note that our four-qubit entanglement monotones fit nicely into the scheme of [9] generating a class of n -qubit entanglement monotones based on bipartite decompositions of $\mathcal{H} = \mathbb{C}^{2^n}$. The basic idea followed there was to consider the manifold of subspaces of \mathcal{H} , i.e. suitable Grassmannians with the corresponding Plücker coordinates for them. Equivalently, we should consider subspaces of the corresponding projective spaces $\mathbf{P}(\mathcal{H})$. Fixing a quadric \mathcal{Q} defined by a bilinear form similar to equation (6) a class of SLOCC invariants expressed in terms of these Plücker coordinates can be generated. In this way we were able to reproduce three of the basic four-qubit invariants (i.e. the triple (H, L, M)). Now we see that by employing also the notion of projective duality all four algebraically independent invariants of the four-qubit case can be written in the Plücker form (see equations (33) and (23) for our new invariants). Note that antisymmetric (Plücker) matrices also appear in the work of V Tapia connected to the problem of constructing multiqubit invariants [22]. It would be nice to establish a connection between his algebraic method and our geometric one.

Finally, let us propose a suggestive geometric picture for four-qubit entanglement. In the usual picture [12] a four-qubit state can be represented by a *single point* in \mathbf{CP}^{15} . Different SLOCC classes correspond to this point lying on different subvarieties in \mathbf{CP}^{15} . Here we would like to suggest an alternative picture. With a four-qubit state we associate a set of *four points* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$, *six lines* $\mathbf{A} \wedge \mathbf{B}, \mathbf{A} \wedge \mathbf{C}, \mathbf{A} \wedge \mathbf{D}, \mathbf{B} \wedge \mathbf{C}, \mathbf{B} \wedge \mathbf{D}, \mathbf{C} \wedge \mathbf{D}$ and *four planes* $\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}, \mathbf{A} \wedge \mathbf{C} \wedge \mathbf{D}, \mathbf{A} \wedge \mathbf{B} \wedge \mathbf{D}, \mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D}$ in the space \mathbf{CP}^3 of smaller dimension. It is easy to see by looking at the intersection properties of these geometrical objects that they form a *tetrahedron* in \mathbf{CP}^3 . This correspondence between entangled states and geometric objects (unlike the previous one) is *nonlocal*. The invariants (I_1, I_2, I_3, I_4) we have proposed obviously characterize the properties of this tetrahedron. For example, for $I_4 = 0$ the four points corresponding to the four vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and \mathbf{D} in \mathbb{C}^4 are not linearly independent (some of them are proportional); hence, the tetrahedron degenerates to a triangle or a line etc depending on the degree of degeneracy. We conjecture that the vanishing of the other three invariants somehow characterizes more intricate degeneracies occurring with lines and planes of the tetrahedron. The class containing no degeneracy is the G_{abcd} -class that can be represented by a regular tetrahedron. It would be interesting to understand how the SLOCC classes arise in this picture.

Let us elucidate the meaning of the proposed correspondence a little bit further. The usual geometric classification schemes for multiqubit systems are based on the use of hyperdeterminants of more general type. These hyperdeterminants describe geometrically hypersurfaces projectively dual to the so-called Segre embedding [12, 20] representing the subvariety of totally separable states [18]. For two, three and four qubits, for instance, the manifold of totally separable states is $\mathbf{CP}^1 \times \mathbf{CP}^1$ embedded in \mathbf{CP}^3 , $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$ in \mathbf{CP}^7 and $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$ in \mathbf{CP}^{15} , respectively. Here $n = 2, 3, 4$ qubit states carrying entanglement are represented by *points* off the Segre surfaces. Such surfaces are representing totally separable states in *different* projective spaces (\mathbf{CP}^{2^n-1}). Here following the spirit of our previous set of papers [9, 16, 17, 23] we prefer to suggest a unified \mathbf{CP}^3 picture. For

$n = 2, 3, 4$ we take \mathbf{CP}^3 with a fixed quadric \mathcal{Q} based on our choice of bilinear form (6). Points lying on \mathcal{Q} as a subvariety of \mathbf{CP}^3 correspond to *null vectors* in \mathbf{C}^4 . For $n = 2$ to a separable or an entangled state corresponds a point *on* or *off* \mathcal{Q} respectively. For $n = 3$ we get the following geometric picture [16, 23]. To a three-qubit state in the *GHZ*-class corresponds a *line* in \mathbf{CP}^3 intersecting \mathcal{Q} at *two* points. To a state in the *W*-class corresponds a line tangent to \mathcal{Q} at a *point*. The separable classes $B(AC)$ and $C(AB)$ are represented by isotropic lines lying entirely in \mathcal{Q} . They are self-dual and anti-self-dual lines belonging to the two different rulings of \mathcal{Q} . The $A(BC)$ and $(A)(B)(C)$ classes again correspond to the degenerate case of points *on* and *off* the quadric \mathcal{Q} (see also the pictorial representation of [22]). We expect a similar pattern to exist also for the four-qubit ($n = 4$) case. Here, we have more lines arranged to form a tetrahedron, and we have to consider constellations of these lines with respect to our fixed quadric \mathcal{Q} . The picture arising in this way has some striking similarity with the Majorana [24] representation of states with spin s . One can represent geometrically a state of spin s as a *single point* in \mathbf{CP}^{2s} , or alternatively, as a *constellation of $2s$ points* on \mathbf{CP}^1 , i.e. the Bloch sphere. Some degeneracies can occur in this case e.g. when $2s$ points degenerate to a single one with multiplicity $2s$ corresponding to the states of highest and lowest weights. In the same spirit we would rather represent n -qubit entangled states in \mathbf{CP}^3 . Here, however, in order to account for the nonlocality of multiqubit quantum entanglement instead of merely a collection of points we also have to consider constellations of lines and planes in \mathbf{CP}^3 . Though this analogy is very appealing we expect it to run out of steam for the $n = 5$ case (five qubits) where probably we should furnish \mathbf{CP}^3 with more extra structures than a quadric. However, for $n \geq 5$, in principle, we can consider constellations of ‘simplexes’ in \mathbf{CP}^3 related to fundamental invariants of the SLOCC group whose combinatorial variability should somehow correspond to the proliferating number of entanglement classes.

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References

- [1] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245
- [2] Coffman V, Kundu J and Wootters W K 2000 *Phys. Rev. A* **61** 052306
- [3] Wong A and Christensen N 2001 *Phys. Rev. A* **63** 044301
- [4] Meyer D A and Wallach N R 2001 *J. Math. Phys.* **43** 4273
- [5] Luque J-G and Thibon J-Y 2003 *Phys. Rev. A* **67** 042303
- [6] Emary C 2004 *J. Phys. A: Math. Gen.* **37** 8293
- [7] Osterloh A and Siewert J 2005 *Phys. Rev. A* **72** 012337
- [8] Luque J-G and Thibon J-Y 2005 *Preprint quant-ph/0506058*
- [9] Lévay P 2005 *J. Phys. A: Math. Gen.* **28** 9075–85
- [10] Dür W, Vidal G and Cirac J I 2000 *Phys. Rev. A* **62** 062314
- [11] Verstraete F, Dehaene J, De Moor B and Verschelde H 2002 *Phys. Rev. A* **65** 052112
- [12] Miyake A 2003 *Phys. Rev. A* **67** 012108
- [13] Bengtsson I, Brännlund J and Życzkowski K 2002 *Int. J. Mod. Phys. A* **17** 4675
- [14] Brody D C and Hughston L P 2001 *J. Geom. Phys.* **38** 19
- [15] Mosseri R and Dandoloff R 2001 *J. Phys. A: Math. Gen.* **34** 10243
- [16] Lévay P 2005 *Phys. Rev. A* **71** 012334
- [17] Lévay P, Nagy Sz and Pipek J 2005 *Phys. Rev. A* **72** 022302
- [18] Heydari H and Björk G 2005 *J. Phys. A: Math. Gen.* **38** 3203–11
- [19] Bennett C, Popescu S, Rohrlich D, Smolin J and Thapliyal A 2001 *Phys. Rev. A* **63** 012307

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- [20] Gelfand I M, Kapranov M M and Zelevinsky A V 1994 *Discriminants, Resultants, and Multidimensional Determinants* (Boston, MA: Birkhäuser)
 - [21] Grier D and Huggins P 2006 *Preprint* [math.CO/0602149](#)
 - [22] Tapia V 2005 *Revista Colombiana de Matemáticas* **39** 37–55
Tapia V 2002 *Preprint* [math-ph/0208010](#)
 - [23] Lévy P 2006 *Preprint* [hep-th/0603136](#)
 - [24] Majorana E 1932 *Nuovo Cimento* **9** 43–50
 - [25] Verstraete F, Dehaene J and De Moor B 2003 *Phys. Rev. A* **68** 012103
 - [26] Acín A, Andrianov A, Jané E and Tarrach R 2001 *J. Phys. A: Math. Gen.* **34** 6725–39